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Three-Dimensional Integrable Models and Associated Tangle Invariants

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In this paper we show that the Boltzmann weights of the three-dimensional Baxter–Bazhanov model give representations of the braid group if some suitable spectral limits are taken. In the trigonometric case we classify all possible spectral limits which produce braid group representations. Furthermore, we prove that for some of them we get cyclotomic invariants of links and for others we obtain tangle invariants generalizing the cyclotomic ones.

KEY WORDS: Three-dimensional solvable models; Zamolodchikov model; Baxter–Bazhanov models; generalized chiral Potts models; Yang–Baxter equation; tetrahedron equation; braid group representation; Markov trace; cyclotomic knot invariants.

1. INTRODUCTION

Baxter and Bazhanov[1] introduced a particularly interesting three-dimensional integrable model with N local states. It is one of the few solvable three-dimensional models and seems to be highly nontrivial.

The Baxter–Bazhonov model is a generalization of the Zamolodchikov model,[2,3] which is the particular case $N=2$. Kashaev *et al.*[4,5] proved that the Boltzmann weights of the Baxter–Bazhanov model satisfy the tetrahedron equations.[2,6,7] This is a generalization of the result obtained by Baxter[8] for the Zamolodchikov model. They use the symmetry properties[4] of the Boltzmann weights, which have been found independently also by Bazhanov and Baxter.[9]

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One of the most important features[1] of the Baxter–Bazhanov model is that apart from a modification of the boundary conditions, it can be obtained as a three-dimensional interpretation of the generalized $sl(n)$-chiral Potts model.[10–12]

Given a two-dimensional integrable model, which has Boltzmann weights satisfying the Yang–Baxter equation, it is an interesting question to ask which braid group representations and hence which link invariants arise therefrom. Akutsu, Deguchi, and Wadati[13,14] invented a general procedure to study this problem and obtained link invariants from most two-dimensional integrable models. Date *et al.*[15] studied the braid group representations and the corresponding (cyclotomic) link invariants arising from the $sl(n)$-chiral Potts model in the trigonometric limit. Following a suggestion made by Jones,[16] they generalized the results of Kobayashi *et al.*[17] for the $sl(2)$-chiral Potts model. The connection of such invariants with the Seifert matrix has been studied by Goldschmidt and Jones.[18]

Following a similar scheme, we study the three-dimensional integrable Baxter–Bazhanov model from the point of view of the link theory. We generalize the results of ref. 15 to the R-matrix with spectral parameters associated to the Baxter–Bazhanov model. We show that, choosing suitable limits of the spectral parameters, this matrix gives cyclic representations of the braid group. In the trigonometric case we classify all possible spectral limits which produce braid group representations. We prove that for some of them we get cyclotomic link invariants, while for other limits of the rapidity variables (spectral parameters) the R-matrix of the Baxter–Bazhanov model gives tangle invariants. Such invariants are generalizations of the cyclotomic invariants previously mentioned.[4]

2. THE THREE-DIMENSIONAL BAXTER–BAZHANOV MODEL AND ITS TWO-DIMENSIONAL REDUCTION

The Baxter–Bazhanov model[1] is an integrable three-dimensional IRF (interaction-round-a-face) model. This means that it is defined on a simple cubic lattice $\mathscr{L}$ and that a spin variable σ is placed at each site of $\mathscr{L}$. From the point of view of statistical mechanics the Baxter–Bazhanov model depends on two integer parameters N ($N \geqslant 2$) and n. Here N is the number of values that each spin σ can take, while n is one of the lattice dimensions (number of elementary cubes in a fixed direction, e.g., in front-to-back direction).

[4] This work is mainly based on the thesis of B. L. Cerchiai (Modelli di Baxter–Bazhanov e di Potts chirale e teoria dei nodi, academic year 1992–93), in fulfilment of the requirements for the degree (laurea) in physics.

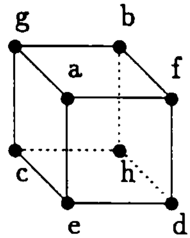

Fig. 1. Elementary cell.

The elementary cube of $\mathscr{L}$ is shown in Fig. 1.

In order to define the Boltzmann weight of the elementary cell shown in Fig. 1 it is necessary to introduce some notation first.

Let x be a complex parameter and k, l, m three integers, $0 \leqslant k, l, m \leqslant N-1$. Let ω be a primitive Nth root of unity

$$\omega = e^{2\pi i/N} \tag{2.1}$$

and

$$\omega^{1/2} = e^{\pi i/N} \tag{2.2}$$

Let Φ and s be the functions defined by

$$\Phi(l) = (\omega^{1/2})^{l(N+l)} \tag{2.3}$$

$$s(k, l) = \omega^{kl} \tag{2.4}$$

Notice that

$$s(k+N, l) = s(k, l+N) = s(k, l) \tag{2.5}$$

$$s(k+l, m) = s(k, m)\, s(l, m) \tag{2.6}$$

$$\Phi(k+l) = \Phi(k)\, \Phi(l)\, s(k, l) \tag{2.7}$$

Moreover, let $w(x, l)$ be the function defined by

$$\frac{w(x, l)}{w(x, 0)} = [\Delta(x)]^l \prod_{k=1}^{l} (1 - \omega^k x)^{-1} \tag{2.8}$$

where

$$\Delta(x) = (1 - x^N)^{1/N} \tag{2.9}$$

With this definition the function $w(x, l)$ is fixed up to the overall normalization factor $w(x, 0)$, while Δ is fixed up to the choice of a phase when taking the root. In particular it is possible to impose the following condition on $w(x, 0)$:

$$w(x, 0) = w(\omega^k x, 0) \tag{2.10}$$

Applying this condition (2.10) to the definition of w in (2.8), it follows immediately that

$$w(x, 0)\, w(x, l+k) = w(x, k)\, w(\omega^k x, l) \tag{2.11}$$

Having introduced all this notation, following ref. 1, the Boltzmann weight of the elementary cube shown in Fig. 1 is constructed as

$$W(a \mid e, f, g \mid b, c, d \mid h) = \sum_{\sigma=0}^{N-1} v_\sigma(a \mid e, f, g \mid b, c, d \mid h) \tag{2.12}$$

with

$$\begin{aligned} &v_\sigma(a \mid e, f, g \mid b, c, d \mid h) \\ &\quad = \frac{w(p'/p, e-c-d+h)}{w(p'/p, a-g-f+b)}\, s(c-h, d-h)\, s(g, a-g-f+b) \\ &\qquad \times \left\{ \frac{w(p/q, d-h-\sigma)\, w(q'/p, \sigma-f+b)\, w(p'/q', a-g-\sigma)}{w(p'/q, e-c-\sigma)[\Phi(a-g-\sigma)]^{-1}} \right. \\ &\qquad \left. \times\, s(\sigma, a-c-f+h) \right\} \end{aligned} \tag{2.13}$$

The parameters p, p', q, q' are the so-called spectral parameters. To stress the dependence of W on these parameters, it would be more correct to write

$$W = W[p, p', q, q']$$

Notice that the spins are seen as elements of $\mathbf{Z}_N$. In the expressions (2.12) and (2.13) σ can be interpreted as a spin at the center of the cube. The elementary interactions are shown in Fig. 2.

This means that in fact we are not considering a simple cubic lattice, but a body-centered cubic lattice. Bazhanov and Baxter noticed[1] that the model obtained in this way is an Ising-type model. Thus it turns out that (up to an overall normalization factor, a site-type, edge-type, and face-type

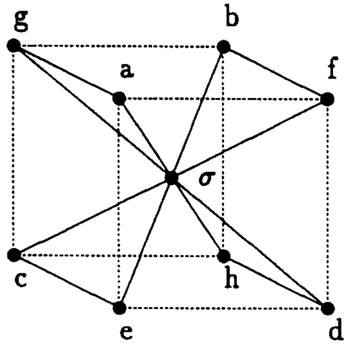

Fig. 2. Interactions in the elementary cube.

equivalence transformation) W satisfies the tetrahedron equation,[8,4,5] which guarantees that the model is integrable:

$$\begin{aligned}
&\sum_d W(a_4 \mid c_2, c_1, c_3 \mid b_1, b_3, b_2 \mid d)\, W'(c_1 \mid b_2, a_3, b_1 \mid c_4, d, c_6 \mid b_4)\\
&\qquad \times W''(b_1 \mid d, c_4, c_3 \mid a_2, b_3, b_4 \mid c_5)\, W'''(d \mid b_2, b_4, b_3 \mid c_5, c_2, c_6 \mid a_1)\\
&\quad = \sum_d W'''(b_1 \mid c_1, c_4, c_3 \mid a_2, a_4, a_3 \mid d)\, W''(c_1 \mid b_2, a_3, a_4 \mid d, c_2, c_6 \mid a_1)\\
&\qquad \times W'(a_4 \mid c_2, d, c_3 \mid a_2, b_3, a_1 \mid c_5)\, W(d \mid a_1, a_3, a_2 \mid c_4, c_5, c_6 \mid b_4)
\end{aligned} \tag{2.14}$$

In this equation $W = W(P)$, $W' = W(P')$, $W'' = W(P'')$, and $W''' = W(P''')$, where

$$\begin{aligned}
P &= (x_1, x_2, x_3, x_4), \qquad & P' &= (x'_1, x'_2, x'_3, x'_4)\\
P'' &= (x''_1, x''_2, x''_3, x''_4), \qquad & P''' &= (x'''_1, x'''_2, x'''_3, x'''_4)
\end{aligned} \tag{2.15}$$

with $(x_1, x_2, x_3, x_4) = (q, q', p, p')$ and the primes are added to the x's in correspondence with primes of the P's. Defining further

$$x_{ij} = x_i \Delta(x_j/x_i) \tag{2.16}$$

we have that the tetrahedron equations (2.14) hold provided that the coordinates of the points P, P', P'', P''' satisfy the following constraints:

$$\begin{aligned}
&\frac{x_2}{x_1} = \frac{x'_2}{x'_1}, \qquad && \frac{x_{12}}{x_1} = \frac{x'_{12}}{x'_1}, \qquad && \frac{x_3}{\omega x_4} = \frac{x'''_2}{x'''_1}\\
&\frac{x_{34}}{\omega^{1/2} x_4} = \frac{x'''_{12}}{x'''_1}, \qquad && \frac{\omega^{1/2} x_{13} x_{24}}{x_{14} x_{32}} = \frac{x''_1}{x''_2}, \qquad && \frac{\omega^{1/2} x_{12} x_{34}}{x_{14} x_{32}} = \frac{x''_{12}}{x''_2}
\end{aligned}$$

$$\frac{x'_{14}x'_{32}}{x'_{13}x'_{24}}=\frac{x''_{14}x''_{32}}{x''_{13}x''_{24}}, \qquad \frac{x'_{12}x'_{34}}{x'_{13}x'_{24}}=\frac{x''_{12}x''_{34}}{x''_{13}x''_{24}}, \qquad \frac{x''_3}{x''_4}=\frac{x'''_3}{x'''_4}$$
$$\frac{x''_{34}}{x''_4}=\frac{x'''_{34}}{x'''_4}, \qquad \frac{x'_4}{x'_3}=\frac{x'''_{13}x'''_{24}}{\omega^{1/2}x'''_{14}x'''_{32}}, \qquad \frac{x'_{34}}{x'_3}=\frac{x'''_{12}x'''_{34}}{x'''_{14}x'''_{32}}$$
$$\frac{\omega^{1/2}x_{32}x'_4x''_{24}x'''_2}{x_3x'_{24}x''_2x'''_{24}}=1, \qquad \frac{x_{13}x'_1x''_{14}x'''_1}{x_1x'_{14}x''_1x'''_{14}}=1$$
$$\frac{x_{14}x'_4x''_{14}x'''_4}{x_4x'_{14}x''_4x'''_{24}}=1, \qquad \frac{\omega^{1/2}x_{13}x'_3x''_{13}x'''_2}{x_3x'_{13}x''_1x'''_{32}}=1 \tag{2.17}$$

At this point it is useful to consider also the Boltzmann weight S of a parallelepiped $\mathscr{P}$ formed by a whole line of n cubes in front-to-back direction with periodic boundary conditions. Let

$$\begin{aligned} \alpha &= (\alpha_1, ..., \alpha_n), \qquad & \beta &= (\beta_1, ..., \beta_n) \\ \gamma &= (\gamma_1, ..., \gamma_n), \qquad & \delta &= (\delta_1, ..., \delta_n) \end{aligned} \tag{2.18}$$

denote the spins on the edges of $\mathscr{P}$ (Fig. 3).

Then

$$S(\alpha, \beta, \gamma, \delta) = \prod_{i \in \mathbf{Z}_n} W(\delta_i \mid \alpha_i, \gamma_i, \delta_{i+1} \mid \gamma_{i+1}, \alpha_{i+1}, \beta_i \mid \beta_{i+1}) \tag{2.19}$$

Notice that S depends only on the pairwise differences of adjacent spins. This means that it is consistent to assume the following equivalence relation between the spins:

$$\alpha \sim \beta \Leftrightarrow \alpha_i - \alpha_{i+1} = \beta_i - \beta_{i+1} \qquad \forall i = 1, ..., n \tag{2.20}$$

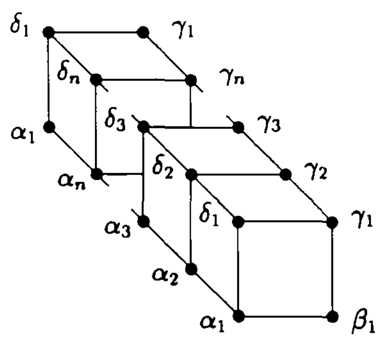

Fig. 3. Parallelepiped $\mathscr{P}$.

Further, following Baxter and Bazhanov, let us introduce also a slightly modified model. Let us substitute the variable σ with the difference of two new spins in front-to-back direction,

$$\sigma = \mu - \mu' \tag{2.21}$$

This means that considering a row of n cubes in front-to-back direction, the following constraint is imposed on the variable σ:

$$\sum_{i \in \mathbf{Z}_n} \sigma_i = 0 \qquad (\mathrm{mod}\ N) \tag{2.22}$$

The model obtained with this change of boundary conditions is called by Baxter and Bazhanov the "modified model." The Boltzmann weight of the parallelepiped $\mathscr{P}$ formed by a line of cubes in front-to-back direction is denoted S_0:

$$S_0(\alpha, \beta, \gamma, \delta) = \prod_{i \in \mathbf{Z}_n} \sum_{\mu_i = 0}^{N-1} v_{\mu_i - \mu_{i+1}}(\delta_i \mid \alpha_i, \gamma_i, \delta_{i+1} \mid \gamma_{i+1}, \alpha_{i+1}, \beta_i \mid \beta_{i+1}) \tag{2.23}$$

with

$$\begin{aligned}
&v_{\mu_i - \mu_{i+1}}(\delta_i \mid \alpha_i, \gamma_i, \delta_{i+1} \mid \gamma_{i+1}, \alpha_{i+1}, \beta_i \mid \beta_{i+1}) \\
&\quad = \frac{w(p'/p, \alpha_i - \alpha_{i+1} - \beta_i + \beta_{i+1})}{w(p'/p, \delta_i - \delta_{i+1} - \gamma_i + \gamma_{i+1})} s(\alpha_{i+1} - \beta_{i+1}, \beta_i - \beta_{i+1}) \\
&\quad \times s(\delta_{i+1}, \delta_i - \delta_{i+1} - \gamma_i + \gamma_{i+1}) \\
&\quad \times \left\{ \frac{w(p/q, \beta_i - \beta_{i+1} - \mu_i + \mu_{i+1})\, w(q'/p, \mu_i - \mu_{i+1} - \gamma_i + \gamma_{i+1})}{w(p'/q, \alpha_i - \alpha_{i+1} - \mu_i + \mu_{i+1})[\Phi(\delta_i - \delta_{i+1} - \mu_i + \mu_{i+1})]^{-1}} \right. \\
&\quad \times w\left(\frac{p'}{q'}, \delta_i - \delta_{i+1} - \mu_i + \mu_{i+1}\right) \\
&\quad \left. \times s(\mu_i - \mu_{i+1}, \delta_i - \alpha_{i+1} - \gamma_i + \beta_{i+1}) \right\}
\end{aligned} \tag{2.24}$$

The key idea of ref. 1 is to describe the Baxter–Bazhanov model as an integrable generalized chiral Potts model[10-12] in the IRF presentation by the prescription in Fig. 4. For this aim, one starts from an edge of the two-dimensional lattice on which the chiral Potts model is defined. This edge is extended in a third additional dimension perpendicular to the plane of the two-dimensional lattice to form a rectangle consisting of n squares. The

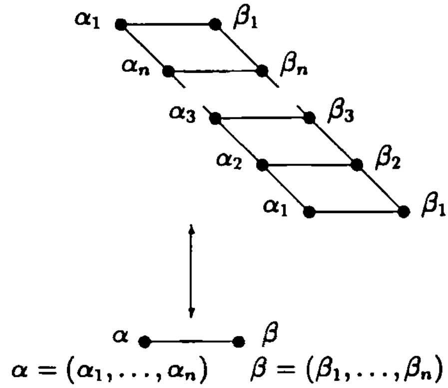

Fig. 4. Reduction procedure.

two spins $\alpha = (\alpha_1, ..., \alpha_n)$ and $\beta = (\beta_1, ..., \beta_n)$ located at the vertices of the two-dimensional lattice are placed on the edges of the rectangle, as shown in Fig. 4. Cyclic boundary conditions are assumed in the new dimension, considering the spins α_1, β_1 as next to α_n, β_n, respectively. Doing this construction for all edges of the two-dimensional lattice, we see that it becomes the three-dimensional cubic lattice $\mathscr{L}$ with N-valued spins at each site.

Baxter and Bazhanov proved that the weight function $W_{pq}(\alpha, \beta)$ of the chiral Potts model associated with an edge can be written in the form

$$\frac{W_{pq}(\alpha, \beta)}{W_{pq}(0, 0)} = \prod_{i=1}^{n} \left\{ \omega^{(\beta_i - \beta_{i+1})(\alpha_{i+1} - \beta_{i+1})} \right.$$
$$\left. \times w\left(\frac{p_i}{q_i}, \alpha_i - \alpha_{i+1} - \beta_i + \beta_{i+1}\right) \right\} \tag{2.25}$$

Notice that the rapidity variables in (2.25) form an n-vector

$$p = (p_1, ..., p_n), \qquad q = (q_1, ..., q_n)$$

exactly as the spins do. In the three-dimensional interpretation the weight W_{pq} is associated to the whole rectangle constructed in Fig. 4. This three-dimensional reinterpretation of the two-dimensional statistical model is allowed by the factorization property (2.25) of the Boltzmann weight: the ith term in the product depends only on the four spins α_i, β_i, α_{i+1}, β_{i+1} lcated at the vertices of the ith elementary square in the rectangle. Notice that not all two-dimensional integrable models have this factorization property.

Let us now consider the star of Fig. 5. Corresponding to this configuration we define the star-Boltzmann weight $W^{(1)}_{\text{star}}$ of the IRF chiral Potts model

$$W^{(1)}_{\text{star}}(p, p', q, q' \mid \alpha, \beta, \gamma, \delta)$$
$$= \sum_{\mu} \frac{W_{p'p}(\alpha, \beta)}{W_{p'p}(\delta, \gamma)} \frac{W_{pq}(\beta, \mu)\, W_{q'p}(\mu, \gamma)\, W_{p'q'}(\delta, \mu)}{W_{p'q}(\alpha, \mu)} \tag{2.26}$$

whose W_{ij} $(i, j = p, p', q, q')$ are the edge-Boltzmann weights defined in (2.25). It turns out that $W^{(1)}_{\text{star}}$ satisfies the Yang–Baxter equation[11,12]

$$\sum_{\sigma} W^{(1)}_{\text{star}}(p, p', q, q' \mid \alpha, \beta, \gamma, \sigma)\, W^{(1)}_{\text{star}}(p, p', r, r' \mid \sigma, \gamma, \delta, \varepsilon)$$
$$\times W^{(1)}_{\text{star}}(q, q', r, r' \mid \alpha, \sigma, \varepsilon, \kappa)$$
$$= \sum_{\sigma} W^{(1)}_{\text{star}}(q, q', r, r' \mid \beta, \gamma, \delta, \sigma)$$
$$\times W^{(1)}_{\text{star}}(p, p', r, r' \mid \alpha, \beta, \sigma, \kappa)\, W^{(1)}_{\text{star}}(p, p', q, q' \mid \kappa, \sigma, \delta, \varepsilon) \tag{2.27}$$

The connection between the chiral Potts model and the Baxter–Bazhanov model arises because it turns out that the Boltzmann weight of the row of cubes in front-to-back direction $\mathscr{P}$ in the modified model exactly coincides with $W^{(1)}_{\text{star}}$,

$$S_0(\alpha, \beta, \gamma, \delta) = W^{(1)}_{\text{star}}(\alpha, \beta, \gamma, \delta) \tag{2.28}$$

Then in order to construct (cyclic) representations of the braid group, the usual procedure[11,15] is to map by a Wu–Kadanoff–Wegener-like transformation the IRF R-matrix defined by $W^{(1)}_{\text{star}}$ to a vertex-type one, and hence to show that it is an intertwiner of the (cyclic) representations of the quantum group $U_q(\hat{gl}_n)$. The main result of this paper is to show in the next sections that by choosing some suitable limits of the spectral

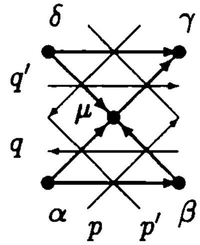

Fig. 5. Elementary star.

parameters characterizing the IRF R-matrix (2.26) of the three-dimensional Baxter–Bazhanov model, one may obtain directly cyclic representations of the braid group, similarly to the two-dimensional case.[13,14]

3. THE CYCLOTOMIC INVARIANTS

In order to construct cyclic representations of the braid group and the related cyclotomic invariants, the starting point is the construction of a $\mathbf{C}^*$-algebra $\mathscr{A}(c)$ and of a functor $\mathscr{F}$ from the category of the uniform oriented tangles $\mathscr{T}_M^M$ to $\mathscr{A}(c)$, following Date *et al.*[15] Notice, however, that they suppose N odd, whereas we consider also the case N even, when this is possible. Let us introduce some notations first.

Let L be a free $\mathbf{Z}_N$ module of rank $n-1$ and suppose it is given by the exact sequence

$$0 \to \operatorname{Ker} \pi = \mathbf{Z}_N(1, \ldots, 1) \to \mathbf{Z}_N^n \to L \to 0$$

This means that it is possible to write the elements of L as

$$\alpha = (\alpha_1, \ldots, \alpha_n)$$

with the equivalence relation (2.20), which implements the $\mathbf{Z}_N^{n-1}$ symmetry of the Baxter–Bazhanov model.

Next let us introduce the nonsingular bilinear form B on L,

$$B(\alpha, \beta) = -\sum_{i \in \mathbf{Z}_n} \alpha_i(\beta_i - \beta_{i+1}) \tag{3.1}$$

which corresponds to the $n \times n$ matrix

$$B_{ij} = \begin{cases} -1 & \text{if} \quad i = j \\ 1 & \text{if} \quad i = j-1 \qquad (\operatorname{mod} n) \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

Let $A(\alpha, \beta)$ be twice the skew-symmetric part of $B(\alpha, \beta)$,

$$A(\alpha, \beta) = B(\alpha, \beta) - B(\beta, \alpha) \tag{3.3}$$

These definitions are consistent, since B respects the equivalence relation (2.20). Notice that the form B considered in §4 of ref. 15 is twice the form B introduced here. The following $\mathbf{Z}_N$-linear isomorphisms $^\vee$ and $^\wedge$ of L can be constructed:

$$\begin{aligned} B(\alpha, \hat{\beta}) &= -B(\beta, \alpha) \\ B(\check{\alpha}, \beta) &= -B(\beta, \alpha) \end{aligned} \tag{3.4}$$

Clearly $^\wedge$ is the inverse of $^\vee$.

Next we consider the category of the uniform oriented tangles $\mathscr{T}^M_M$. Recall that this category is defined as follows.[19,20] The objects of $\mathscr{T}^M_M$ are given by configurations c of M strings. By a configuration we mean a map c: $\{1, ..., M\} \mapsto \{\pm 1\}$ that we write $c = (c(1), ..., c(M))$. We say that c is of type (M_+, M_-), if $M_+ =$ number of k for which $c(k) = 1$, $M_- =$ number of k for which $c(k) = -1$. Obviously $M = M_+ + M_-$. Graphically a configuration c is a set of M strings such that the kth string, counted from left to right, is downward or upward according as $c(k) = 1$ or $c(k) = -1$, respectively. There is an action of the symmetric group S_M on the set of configurations of M strings, and it is defined by

$$s(c) = c \circ s^{-1}, \qquad \text{with} \quad s \in S_M$$

Notice that $s(c) = c'$ if and only if c and c' are of the same type.

An (M, M)-tangle t is a smooth one-dimensional compact submanifold of $\mathbf{R}^2 \times [0, 1]$ such that

$$\partial t = t \cap (\mathbf{R}^2 \times \{0, 1\}) = \{(i, 0, 0) \mid i = 1, ..., M\} \cup \{(j, 0, 1) \mid j = 1, ..., M\}$$

and such that every boundary point is orthogonal to the planes $\mathbf{R} \times 0$ and $\mathbf{R} \times 1$. A tangle is said to be oriented if the manifold is oriented. Two configurations c and c' are associated to each tangle in such a way that $c(k) = \pm 1$ if the unit tangent vector to t in $(k, 0, 1)$ is $(\pm 1, 0, 0)$ and $c'(k) = \pm 1$ if the unit tangent vector in $(k, 0, 0)$ is $(\pm 1, 0, 0)$, respectively. Two tangles are isotopic if there exists an isotopy of the strip $\mathbf{R}^2 \times [0, 1]$ to itself, which is the identity on the boundary and which carries one tangle into the other. An (M, M)-tangle is said to be uniform if it intersects each horizontal line between the top edge and the bottom edge at exactly M, $M-1$, or $M-2$ points. In the case c and c' are of the same type, the morphisms $\hom(c, c')$ of the category of the uniform oriented tangles $\mathscr{T}^M_M$ are given by the set of the isotopy classes of tangles with c on the top and c' on the bottom, otherwise $\hom(c, c') = \emptyset$.

Now, let us introduce the algebra $\mathscr{A}(c)$. It is a $\mathbf{C}$-algebra with a unit 1. If $M = 0$ or $M = 1$, $\mathscr{A}(c)$ is simply $\mathbf{C}$. If $M \geqslant 2$, $\mathscr{A}(c)$ is the algebra with generators $x^\alpha_k = x^\alpha_k(c)$ where $1 \leqslant k \leqslant M-1$ and $\alpha \in L$. We impose the relations

$$\begin{aligned}
x^0_k &= 1 \\
x^\alpha_k x^\beta_k &= \omega^{A(\beta,\alpha)/2} x^{\alpha+\beta}_k && \text{if} \quad (c(k), c(k+1)) = (1, 1) \\
&= \omega^{A(\alpha,\beta)/2} x^{\alpha+\beta}_k && \text{if} \quad (c(k), c(k+1)) = (-1, -1) \\
&= x^{\alpha+\beta}_k && \text{if} \quad c(k) \neq c(k+1)
\end{aligned}$$

$$
\begin{aligned}
x_k^\alpha x_{k+1}^\beta &= \omega^{B(\beta,\alpha)} x_{k+1}^\beta x_k^\alpha && \text{if} \quad c(k+1) = -1 \\
&= \omega^{\beta(\alpha,\beta)} x_{k+1}^\beta x_k^\alpha && \text{if} \quad c(k+1) = 1 \\
x_k^\alpha x_{k'}^\beta &= x_{k'}^\beta x_k^\alpha && \text{if} \quad |k-k'| \geqslant 2
\end{aligned}
\tag{3.5}
$$

On the algebra $\mathscr{A}(c)$ there is a linear involution, which is defined by its action on the generators

$$
(x_k^\alpha)^* = x_k^{-\alpha} \tag{3.6}
$$

After introducing the algebras $\mathscr{A}(c)$, Date *et al.* showed that two algebras $\mathscr{A}(c)$ and $\mathscr{A}(c')$ relative to configurations c and c' of the same type are canonically isomorphic if N is an odd number:

$$
i_{c'}^c \colon \mathscr{A}(c) \to \mathscr{A}(c') \tag{3.7}
$$

To see this, let us consider the case that $c' = s_k(c)$ with s_k the permutation $(k, k+1)$. If $c(k) = c(k+1)$, then $c = c'$ and $i_{c'}^c = id_{\mathscr{A}(c)}$. Otherwise, in the case that $c(k) = -1$, $c(k+1) = 1$, $i_{c'}^c$ is defined as follows:

$$
\begin{aligned}
i_{c'}^c(x_{k'}^\alpha(c)) &= x_k^\alpha(c') && \text{if} \quad |k-k'| \geqslant 2 \\
i_{c'}^c(x_{k\pm 1}^\alpha(c)) &= x_{k\pm 1}^\alpha(c')\, x_k^{(\alpha+\dot\alpha)/2}(c') \\
i_{c'}^c(x_k^\alpha(c)) &= x_k^{-\dot\alpha}(c')
\end{aligned}
\tag{3.8}
$$

while in the case that $c(k) = 1$, $c(k+1) = -1$,

$$
\begin{aligned}
i_{c'}^c(x_{k'}^\alpha(c)) &= x_k^\alpha(c') && \text{if} \quad |k-k'| \geqslant 2 \\
i_{c'}^c(x_{k\pm 1}^\alpha(c)) &= x_{k\pm 1}^\alpha(c')\, x_k^{(\alpha+\dot\alpha)/2}(c') \\
i_{c'}^c(x_k^\alpha(c)) &= x_k^{-\dot\alpha}(c')
\end{aligned}
\tag{3.9}
$$

In terms of the operators x_k^α it is possible to define the operators describing the images of the functor $\mathscr{F}$ of the elementary tangles as

$$
\begin{aligned}
T_k(c) &= \frac{1}{\sqrt{D}} \sum_{\alpha \in L} \omega^{-B(\alpha,\alpha)/2} x_k^\alpha(c) && \text{if} \quad c(k) = c(k+1) \\
&= \frac{1}{D} \sum_{\alpha,\beta \in L} \omega^{B(\beta,\beta)/2 + B(\alpha,\beta)} x_k^\alpha(c) && \text{if} \quad c(k) \neq c(k+1), \qquad N \text{ odd} \\
E_k(c) &= \frac{1}{\sqrt{D}} \sum_{\alpha \in L} x_k^\alpha(c) && \text{if} \quad c(k) \neq c(k+1), \qquad N \text{ odd}
\end{aligned}
\tag{3.10}
$$

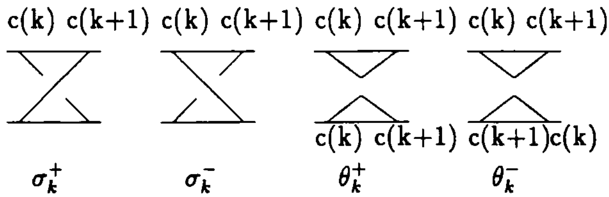

Fig. 6. Elementary tangles.

where $D = N^{n-1}$. Notice that if N is even, we consider only the case $c(k) = 1\ \forall k$ or $c(k) = -1\ \forall k$, in which the tangles reduce to ordinary braids. In that case the division by 2 in the exponent is not interpreted as an operation in $\mathbf{Z}_N$, but as taking a square root of ω. The value of the cyclotomic knot invariant introduced in (3.21) is independent of the choice of the root. Notice also that E_k is defined only when $c(k) \neq c(k+1)$, N odd, and that whenever this element will be considered these conditions are implied. Recall that the functor $\mathcal{F}$ from the category $\mathcal{T}_M^M$ of the uniform oriented tangles is constructed as follows.[15] First, the morphisms of $\mathcal{T}_M^M$ are generated by the elementary tangles shown in Fig. 6.

Let us suppose N odd. As a consequence of the defining relations (3.10) of the morphisms T_k, E_k and using the commutation relations (3.5) of the x_k^α, Date *et al.*[15] verified that the elements (3.10) satisfy the following relations, in which the strings should be oriented in all possible ways:

$$\begin{aligned}
T_k^* &= T_k^{-1} \\
T_k T_{k'} &= T_{k'} T_k \qquad \text{if} \quad |k-k'| \geqslant 2 \\
T_k T_{k+1} T_k &= T_{k+1} T_k T_{k+1} \\
E_k^* &= E_k \\
E_k T_{k'} &= T_{k'} E_k \qquad \text{if} \quad |k-k'| \geqslant 2 \\
E_k E_{k'} &= E_{k'} E_k \qquad \text{if} \quad |k-k'| \geqslant 2 \\
T_k^{\pm 1} E_k &= E_k \\
E_k E_{k\pm 1} E_k &= E_k \\
E_k E_{k\pm 1} T_k^{\pm 1} &= E_k T_{k\pm 1}^{\mp 1}
\end{aligned} \tag{3.11}$$

Notice that the first relation means that T_k is unitary, while the second and the third are the braid group relations. As an example, let us show the third relation, if $c(k)=1\ \forall k$. The left-hand side of the equation may be written as

$$
\begin{aligned}
T_k T_{k+1} T_k &= \sum_{\alpha,\beta,\gamma} \omega^{-1/2[B(\alpha,\alpha)+B(\beta,\beta)+B(\gamma,\gamma))} x_k^{\alpha} x_{k+1}^{\beta} x_k^{\gamma} \\
&= \sum_{\alpha,\beta,\gamma} \omega^{-1/2[B(\alpha,\alpha)+B(\beta,\beta)+B(\gamma,\gamma)+A(\alpha,\gamma)+2B(\gamma,\beta)]} x_k^{\alpha+\gamma} x_{k+1}^{\beta} \\
&= \sum_{\beta,\gamma,\delta} \omega^{-1/2[B(\delta,\delta)+B(\beta,\beta)+2B(\gamma,\gamma)-2B(\gamma,\delta)+2B(\gamma,\beta)]} x_k^{\delta} x_{k+1}^{\beta}
\end{aligned}
$$

By the same arguments the right-hand side may be written as

$$
\begin{aligned}
T_{k+1} T_k T_{k+1} &= \sum_{\alpha,\delta,\gamma} \omega^{-1/2[B(\alpha,\alpha)+B(\delta,\delta)+B(\gamma,\gamma))} x_{k+1}^{\gamma} x_k^{\delta} x_{k+1}^{\alpha} \\
&= \sum_{\beta,\gamma,\delta} \omega^{-1/2[B(\delta,\delta)+B(\beta,\beta)+2B(\gamma,\gamma)+2B(\delta,\gamma)-2B(\beta,\gamma)]} x_k^{\delta} x_{k+1}^{\beta}
\end{aligned}
$$

Making the change of variables $\gamma \to \hat{\gamma}$, we obtain exactly the same expression, which gives the left-hand side.

The relations (3.11) are the defining relations of the class of the morphisms $\hom(c,c)$ in the category $\mathscr{T}_M^M$. This means that the functor $\mathscr{F}$ mapping $\mathscr{T}_M^M$ to $\mathscr{A}(c)$ defined by

$$
\begin{gathered}
\mathscr{F}(c)=\mathscr{A}(c) \\
\mathscr{F}(\sigma_k^{\pm})=\mathscr{M}(T_k(c)^{\pm 1}), \qquad \mathscr{F}(\theta_k^{\pm})=\mathscr{M}(E_k(c)^{\pm 1})
\end{gathered}
\tag{3.12}
$$

is well defined. Here $\mathscr{M}(a)\in \mathrm{End}(\mathscr{A}(c))$ denotes the left multiplication by $a\in\mathscr{A}$.

Notice that if $c(k)=1\ \forall k$ or $c(k)=-1\ \forall k$, then the tangles T_k give a representation of the ordinary braid group. In that case it is possible to generalize the construction to N even, and it is immediate to see that the first, second, and third of relations (3.11) remain valid.

Moreover, in the case that $c(k)=1\ \forall k$ or $c(k)=-1\ \forall k$, we can consider also the right multiplication by elements of $\mathscr{A}(c)$ and we obtain a right-regular representation of the braid group, not only a left-regular one. Further, if $n=2$ and $c(k)=1\ \forall k$ or $c(k)=-1\ \forall k$, the operators T_k satisfy the "generalized skein relations"

$$
T_k^l=\sum_{i=0}^{l-1} A_i T_k^i \tag{3.13}
$$

with the order l of the skein relation given by

$$l=\begin{cases}(N+2)/2 & \text{if } N \text{ even} \\ (N+1)/2 & \text{if } N \text{ odd}\end{cases} \tag{3.14}$$

Formula (3.14) can be valid only when the equation

$$x^2=1 \qquad (\mathrm{mod}\, N) \tag{3.15}$$

admits no other solutions apart from the two values $x=1$ (mod N) and $x=N-1$ (mod N). In particular, it is valid for the prime numbers. It shows that for $n=2$ and $N=2, 3$ the algebra defined by the operators T_k can be expressed in terms of the Hecke algebra.[13,14] More generally, operators satisfying generalized skein relations like (3.13) can be obtained with a "cabling procedure" starting from the generators of the Hecke algebra. In Eq. (3.13) the coefficients A_i are the solutions of the linear system

$$\frac{1}{D^{l/2}} \sum_{\alpha(2),\dots,\alpha(l)\in L} \prod_{i=1}^{l-1} \omega^{B(\alpha(i+1),\alpha(i)-\alpha(i+1))}$$
$$=\delta_{\alpha(1),(0,0)}A_0+\sum_{r=1}^{l-1}\frac{A_r}{D^{r/2}} \sum_{\alpha(2),\dots,\alpha(r)\in L} \prod_{i=1}^{r-1} \omega^{B(\alpha(i+1),\alpha(i)-\alpha(i+1))} \tag{3.16}$$

Moreover, let Tr denote the usual matrix trace on $\mathrm{End}(\mathscr{A}(c))$ normalized as $\mathrm{Tr}(1)=1$.

With all these preliminaries an invariant of oriented links can be constructed as follows. Let $\hat{t}$ denote the link obtained by closing the tangle t, and let v be the number of its crossings. Further, let (k, i), $1\leqslant i\leqslant m_k$, denote the ith crossing, counted from top to bottom, between the strings k, $k+1$. The sign of such a crossing is denoted $\varepsilon(k, i)$, and we say that $(k, i)<(k', i')$ if the crossing (k, i) is above (k', i') in the diagram. Then

$$\mathscr{F}(t)=\mathscr{M}(T_{k_1}^{\varepsilon(k_1,1)}\cdots T_{k_v}^{\varepsilon(k_v,m_{k_v})}) \tag{3.17}$$

and the expression

$$\tau(\hat{t})=D^{(M-1)/2}\,\mathrm{Tr}(\mathscr{M}(T_{k_1}^{\varepsilon(k_1,1)}\cdots T_{k_m}^{\varepsilon(k_v,m_{k_v})})) \qquad \text{for} \quad t\in \mathrm{hom}(c,c) \tag{3.18}$$

gives an invariant of oriented links. Further, the Tr can be defined by its action on the generators

$$\mathrm{Tr}(x_k^{\alpha_1}\cdots x_k^{\alpha_n})=\begin{cases}1 & \text{if} \quad \alpha_1=\cdots\alpha_n=0 \\ 0 & \text{otherwise}\end{cases} \tag{3.19}$$

The quantity (3.18) is invariant under the Markov moves

$$\begin{aligned} \tau(\widehat{tt'}) &= \tau(\widehat{t't}) \qquad t \in \hom(c, c'),\ t' \in \hom(c', c) \\ \tau(\widehat{t\theta_M^{+}}) &= \tau(\hat{t}) \qquad t \in \hom(c, c) \\ \tau(\widehat{t\sigma_M^{\pm}}) &= \tau(\hat{t}) \qquad t \in \hom(c, c) \end{aligned} \tag{3.20}$$

where c, c' are configurations of M strings, and $c(M+1)=c(M)$. Because of (3.20), the expression (3.18) gives a cyclotomic invariant of the tangle. Here by cyclotomic invariant we mean a link invariant defined through a cyclic representation of the braid group. In the case that the tangles are associated to braids, the invariant is well defined also for even values of N. Moreover, Date *et al.* have shown that if $\hat{b}$ is a closure of a braid $b \in B_M$ with v crossings, then (3.18) becomes

$$\tau(\hat{b}) = N^{(n-1)(M-v-1)/2} \sum_{\alpha \in \mathbf{Z}_N^{v-M+1} \otimes \mathbf{Z}_N^{n-1}} \omega^{Q(\alpha,\alpha)/2}, \qquad \text{with} \quad b \in B_M \tag{3.21}$$

Here $Q(\alpha, \alpha)$ is the bilinear form determined by the matrix

$$Q = S \otimes B' + S^{\mathrm{T}} \otimes B'^{\mathrm{T}} \tag{3.22}$$

where S is a $(v-M+1) \times (v-M+1)$ Seifert matrix for $\hat{b}$; B' is the $n-1 \times n-1$ matrix given by $B'_{ij}=1$ if $i \leqslant j$, $B'_{ij}=0$ otherwise; and T denotes the transposition of a matrix. The matrix B' is associated with the same quadratic form given by B, because it is the obtained by making the change of the basis $\alpha_i \to \alpha_i - \alpha_{i+1}$ for $i=1, ..., n-1$ in the module L. Notice that it is not necessary to consider $\alpha_n - \alpha_1$, because the rank of L is only $n-1$ by (2.20).

To obtain (3.21), observe that by using the definition of the operators T_k in (3.10), the commutation relations (3.5), and the expression (3.19) for the trace, we obtain

$$\begin{aligned} \tau(\hat{t}) = D^{M-v-1} \sum_{\alpha(1,1),\dots,\alpha(v,m_v-1) \in L} & \Big\{ \omega^{1/2 \sum_{k,i} [-(\varepsilon(k,i)+\varepsilon(k,i+1)) B(\alpha(k,i),\alpha(k,i))]} \\ & \times \omega^{1/2 \sum_{k,i} [(1+\varepsilon(k,i+1)) B(\alpha(k,i),\alpha(k,i+1)) + (\varepsilon(k,i+1)-1) B(\alpha(k,i+1),\alpha(k,i))]} \\ & \times \omega^{\sum_{k,i} [B(\alpha(k,i),\alpha(k+1,i^*)) - B(\alpha(k,i-1),\alpha(k+1,i^*))]} \Big\} \end{aligned} \tag{3.23}$$

Here i^* signifies the largest j such that $(k+1, j) < (k, i)$. Date *et al.*[15] have shown that it is possible to construct a Seifert surface for $\hat{t}$ by means of the Seifert algorithm, so that the corresponding Seifert form ϕ is given by

$$\begin{aligned}
&\phi(\gamma(k,i),\gamma(k,i)) \\
&\qquad = -(\varepsilon(k,i)+\varepsilon(k,i+1))/2 \\
&\phi(\gamma(k,i),\gamma(k,i+1)) \\
&\qquad = (1+\varepsilon(k,i+1))/2 \\
&\phi(\gamma(k,i+1),\gamma(k,i)) \\
&\qquad = (-1+\varepsilon(k,i+1))/2 \\
&\phi(\gamma(k,i),\gamma(k+1,j)) \\
&\qquad = 1 \qquad \text{if} \quad (k+1,j)<(k,i)<(k+1,j+1)<(k,i+1) \\
&\phi(\gamma(k,i),\gamma(k+1,j)) \\
&\qquad = -1 \qquad \text{if} \quad (k,i)<(k+1,j)<(k,i+1)<(k+1,i+1)
\end{aligned} \tag{3.24}$$

In this equation $\gamma(k, i)$ is the cycle passing counterclockwise through the crossings (k, i) and $(k, i+1)$. Using this Seifert form, it is immediate to see that (3.23) coincides with the expression (3.21).

Now (3.22) has a topological meaning, since Q is a presentation matrix for the $\mathbf{Z}_N$ module $H_1(M_n, \mathbf{Z}_N)$. Here M_n is the nth cyclic covering of S^3 branched along the link $\hat{b}$. This means that for N an odd prime number the module of $\tau(\hat{b})$ can be written as

$$|\tau(\hat{b})| = N^{\beta_n/2} \tag{3.25}$$

where β_n is the first Betti number of M_n relative to the homology group $H_1(M_n, \mathbf{Z}_N)$. Hence, if the quadratic from B' is nonsingular, τ can be expressed as a function of products of classical Alexander polynomials associated to the link $\hat{b}$.

4. THE SPECTRAL LIMITS OF THE IRF 2D REDUCED BAXTER–BAZHANOV MODEL *R*-MATRIX AND THE TANGLE INVARIANTS

In this section we shall show that it is possible to obtain directly the cyclotomic invariants from the Boltzmann weights S of the 3D Baxter–Bazhanov model (see Section 2), after taking some suitable limits of the spectral parameters (p, p', q, q'). Furthermore, we shall show that taking other limits of the spectral parameters, it is possible to obtain generalizations of the cyclotomic invariants from the Boltzmann weights S_0 of the modified Baxter–Bazhanov model. The first step is to define the Yang–Baxter operators

$$
\begin{aligned}
&(Y_{k0}(p, p', q, q'))_{\alpha(1)\cdots\alpha(M-1)}^{\alpha'(1)\cdots\alpha'(M-1)} \\
&\quad = \frac{1}{D}\left(\prod_{l \neq k} \delta_{\alpha(l)\alpha'(l)}\right) S_0(\alpha(k-1), \alpha(k), \alpha(k+1), \alpha'(k)) \\
&(Y_k(p, p', q, q'))_{\alpha(1)\cdots\alpha(M-1)}^{\alpha'(1)\cdots\alpha'(M-1)} \\
&\quad = \frac{1}{D}\left(\prod_{l \neq k} \delta_{\alpha(l)\alpha'(l)}\right) S(\alpha(k-1), \alpha(k), \alpha(k+1), \alpha'(k))
\end{aligned}
\tag{4.1}
$$

These operators act on a subspace $(\mathscr{W}^{(0)})^{\otimes M-1} \subset \mathscr{W}^{\otimes M-1}$, where $\mathscr{W} = (C^N)^{\otimes n}$ and $\mathscr{W}^{(0)}$ is the subspace generated by the elements of $\mathscr{W}$,

$$
\xi_\alpha = \sum_k \varepsilon_{\alpha_1 + k} \otimes \cdots \otimes \varepsilon_{\alpha_n + k} \tag{4.2}
$$

where ε_i is the canonical base of C^N and $\alpha \in L$. The subspace $\mathscr{W}^{(0)}$ has dimension $D = N^{n-1}$, while $\mathscr{W}$ has dimension N^n. But this restriction is necessary in order to implement the $\mathbf{Z}_N^{n-1}$ symmetry of the Baxter–Bazhanov model and hence the equivalence relation (2.20).

Then Yang–Baxter operators depend on the spectral parameters (p, p', q, q'). In analogy with the standard procedure established by, e.g., Akatsu, Deguchi, and Wadati,[13,14] the operators Y_k and Y_{k0} give a matrix representation of the braid group B_M if some spectral limits on (p, p', q, q') are taken. It turns out that in these limits Y_k goes either to the left-regular or to the right-regular representation of the operators $T_k(c)^{\pm 1}$ with $c(k) = 1\ \ \forall k = 1, ..., M$ or $c(k) = -1\ \ \forall k = 1, ..., M$. To find braid group representations the first thing is to look for the values of the spectral parameters where the model is critical. This means that we must consider the trigonometric limit in which all the elementary cubes in the parallelepiped $\mathscr{P}$ considered in Section 2 have the same spectral parameters. This assumption guarantees that the model is homogeneous. Then we have found the following limits in which we obtain the left-regular representation of the operators $T_k^{\pm 1}$, $k = 1, ..., M-1$:

$$
\begin{aligned}
&\text{(Ia)} \quad p \ll q \ll p' = q'\colon Y_k \mapsto T_k(c), && c(k) = 1, \quad \forall k = 1, ..., M \\
&\text{(Ib)} \quad q \ll p \ll p' = q'\colon Y_k \mapsto T_k^{-1}(c), && c(k) = 1, \quad \forall k = 1, ..., M \\
&\text{(IIa)} \quad p' \ll q' \ll p = q\colon Y_k \mapsto T_k(c), && c(k) = -1, \quad \forall k = 1, ..., M \\
&\text{(IIb)} \quad q' \ll p' \ll p = q\colon Y_k \mapsto T_k^{-1}(c), && c(k) = -1, \quad \forall k = 1, ..., M
\end{aligned}
\tag{4.3}
$$

To see this, let us choose the following base of the algebra $\mathscr{A}(c)$,[15]

$$\{y(c) = \omega^{(-1/2)\sum_{k=1}^{M-2} B^{c(k+1)}(\alpha(k),\alpha(k+1))} x_1^{\alpha(1)} \cdots x_{M-1}^{\alpha(M-1)}\}_{\alpha(k)\in L} \tag{4.4}$$

where

$$B^c(\alpha, \beta) = \begin{cases} B(\alpha, \beta) & \text{if} \quad c = 1 \\ B(\beta, \alpha) & \text{if} \quad c = -1 \end{cases} \tag{4.5}$$

The map

$$\rho\colon A(c) \mapsto (\mathscr{W}^{(0)})^{\otimes M-1} \tag{4.6}$$

defined by

$$\rho(y(c)) = \xi_{\alpha(1)} \otimes \cdots \otimes \xi_{\alpha(M-1)} \tag{4.7}$$

is an isomorphism of $\mathbf{C}^*$-algebras. Let us prove that (Ia) is right. The matrix elements of the operators $\rho T_k \rho^{-1}$ in the case $c(k) = 1$ $\forall k$, omitting ρ, can be written as

$$\begin{aligned} &(T_k)_{\alpha(1)\cdots\alpha(M-1)}^{\alpha'(1)\cdots\alpha'(M-1)} \\ &\quad \sim \left(\prod_{l \neq k} \delta_{\alpha(l)\alpha'(l)}\right) \omega^{B(\alpha'(k),\alpha(k))} \frac{1}{\sqrt{D}} \\ &\qquad \times \omega^{[1/2(B(\alpha(k)-\alpha'(k),\alpha(k+1)) - B(\alpha(k-1),\alpha(k)-\alpha'(k)) - B(\alpha(k),\alpha(k)) - B(\alpha'(k),\alpha'(k)))]} \end{aligned} \tag{4.8}$$

where $\sim$ means the isomorphism given by conjugation with ρ. The Yang–Baxter operator Y_k in the limit (Ia) gives the same matrix operators, provided a similarity transformation is made. To obtain this result, let us calculate the limits of the function w defined in (2.8). We find

$$\frac{w(x, l)}{w(x, 0)} = \begin{cases} \Phi(l)^{-1} & \text{if} \quad x \to \infty \\ \delta_{l0} & \text{if} \quad x \to 1 \\ 1 & \text{if} \quad x \to 0 \end{cases} \tag{4.9}$$

Using these limits, it is possible to show that

$$\frac{W_{pq}(\alpha, \beta)}{W_{pq}(0, 0)} = \begin{cases} \omega^{B(\alpha, \alpha-\beta)} & \text{if} \quad p/q \to \infty \\ \delta_{\alpha,\beta} & \text{if} \quad p/q \to 1 \\ \omega^{B(\beta, \alpha-\beta)} & \text{if} \quad p/q \to 0 \end{cases} \tag{4.10}$$

From (2.28) it follows immediately that in the limit (Ia)

$$S(\alpha, \beta, \gamma, \delta) = S_0(\alpha, \beta, \gamma, \delta) = \omega^{B(\delta,\beta) - B(\delta,\delta) + B(\alpha,\delta-\beta)} \tag{4.11}$$

To obtain (4.8) from (4.11), we multiply S by the factor

$$\sqrt{D}\, \omega^{[B(\delta,\delta) - B(\beta,\beta) + B(\beta-\delta,\gamma) + B(\alpha,\beta-\delta)]/2} \tag{4.12}$$

It is a site-type, edge-type, face-type equivalence transformation and does not change the factorization properties nor the partition function of the model. With the same tools it is possible to see that (Ib), (IIa), (IIb) hold, provided that S is multiplied by the factor (4.12) in the case (Ib), and by the factor

$$\sqrt{D}\, \omega^{[B(\beta,\beta) - B(\delta,\delta) - B(\beta-\delta,\alpha) - B(\gamma,\beta-\delta)]/2} \tag{4.13}$$

in the cases (IIa) and (IIb). Further, by the same arguments, it is possible to prove that there are other limits giving the $T_k^{\pm 1}$ in the right-regular representation, obtained from the left-regular one by transposing the matrices. These limits are given by

$$\begin{array}{lll} \text{(IIIa)} & p = q \ll p' \ll q'\colon\ Y_k \mapsto T_k(c), & c(k) = 1, \quad \forall k = 1, ..., M \\ \text{(IIIb)} & p = q \ll q' \ll p'\colon\ Y_k \mapsto T_k^{-1}(c), & c(k) = 1, \quad \forall k = 1, ..., M \\ \text{(IVa)} & p' = q' \ll p \ll q\colon\ Y_k \mapsto T_k(c), & c(k) = -1, \quad \forall k = 1, ..., M \\ \text{(IVb)} & p' = q' \ll q \ll p\colon\ Y_k \mapsto T_k^{-1}(c), & c(k) = -1, \quad \forall k = 1, ..., M \end{array} \tag{4.14}$$

At this point a question arises: it is possible to get other kinds of braid group representations and hence other link invariants starting from the Yang–Baxter equation of the Baxter–Bazhanov model? For N odd we fix the configuration of $2M-1$ strings where $c(2k-1) = -1$, $c(2k) = 1$, $\forall k = 1, ..., M-2$, $c(2M-1) = -1$. We obtain the following picture for $k = 1, ..., M-3$:

$$\begin{array}{ll} \text{(Va)} & p' \ll q' \ll p \ll q\colon\quad Y_{k0} \mapsto \\ & T_{2k+1}(s_{2k+1}(c))\, T_{2k}(s_{2k+1}(c))\, T_{2k+2}(s_{2k+1}(c))\, T_{2k+1}(c)^{-1} \\ \text{(Vb)} & q' \ll p' \ll q \ll p\colon\quad Y_{k0} \mapsto \\ & T_{2k+1}(s_{2k+1}(c))\, T_{2k}(s_{2k+1}(c))^{-1}\, T_{2k+2}(s_{2k+1}(c))^{-1}\, T_{2k+1}(c)^{-1} \\ \text{(VIa)} & p \ll q \ll p' \ll q'\colon\quad Y_{k0} \mapsto \\ & T_{2k+1}(s_{2k+1}(c))^{-1}\, T_{2k}(s_{2k+1}(c))\, T_{2k+2}(s_{2k+1}(c))\, T_{2k+1}(c) \\ \text{(VIb)} & q \ll p \ll q' \ll p'\colon\quad Y_{k0} \mapsto \\ & T_{2k+1}(s_{2k+1}(c))^{-1}\, T_{2k}(s_{2k+1}(c))^{-1}\, T_{2k+2}(s_{2k+1}(c))^{-1}\, T_{2k+1}(c) \end{array}$$

(VIIa) $p \ll q \ll q' \ll p'$: $Y_{k0} \mapsto$
$$T_{2k+1}(s_{2k+1}(c))^{-1}\, T_{2k}(s_{2k+1}(c))\, T_{2k+2}(s_{2k+1}(c))^{-1}\, T_{2k+1}(c)$$

(VIIb) $q \ll p \ll p' \ll q'$: $Y_{k0} \mapsto$
$$T_{2k+1}(s_{2k+1}(c))^{-1}\, T_{2k}(s_{2k+1}(c))^{-1}\, T_{2k+2}(s_{2k+1}(c))\, T_{2k+1}(c)$$

(VIIIa) $q' \ll p' \ll p \ll q$: $Y_{k0} \mapsto$
$$T_{2k+1}(s_{2k+1}(c))\, T_{2k}(s_{2k+1}(c))\, T_{2k+2}(s_{2k+1}(c))^{-1}\, T_{2k+1}(c)^{-1}$$

(VIIIb) $p' \ll q' \ll q \ll p$: $Y_{k0} \mapsto$
$$T_{2k+1}(s_{2k+1}(c))\, T_{2k}(s_{2k+1}(c))^{-1}\, T_{2k+2}(s_{2k+1}(c))\, T_{2k+1}(c)^{-1}$$

(IXa) $p \ll p' \ll q \ll q'$: $Y_{k0} \mapsto$
$$T_{2k+1}(s_{2k+1}(c))\, T_{2k}(s_{2k+1}(c))\, T_{2k+2}(s_{2k+1}(c))\, T_{2k+1}(c)$$

(IXb) $q \ll q' \ll p \ll p'$: $Y_{k0} \mapsto$
$$T_{2k+1}(s_{2k+1}(c))^{-1}\, T_{2k}(s_{2k+1}(c))^{-1}\, T_{2k+2}(s_{2k+1}(c))^{-1}\, T_{2k+1}(c)^{-1}$$

(Xa) $p' \ll p \ll q' \ll q$: $Y_{k0} \mapsto$
$$T_{2k+1}(s_{2k+1}(c))\, T_{2k}(s_{2k+1}(c))\, T_{2k+2}(s_{2k+1}(c))\, T_{2k+1}(c)$$

(Xb) $q' \ll q \ll p' \ll p$: $Y_{k0} \mapsto$
$$T_{2k+1}(s_{2k+1}(c))^{-1}\, T_{2k}(s_{2k+1}(c))^{-1}\, T_{2k+2}(s_{2k+1}(c))^{-1}\, T_{2k+1}(c)^{-1} \tag{4.15}$$

Now we must explain the meaning of the products

$$T_{\pm 2k+1}(s_{2k+1}(c))\, T_{\pm 2k}(s_{2k+1}(c))\, T_{\pm 2k+2}(s_{2k+1}(c))\, T_{\pm 2k+1}(c)$$

where $T_{\pm 1k} = T_k^{\pm 1}$. We construct a representation $\mathscr{R}$ of $A(c)$ on $(\mathscr{W}^{(0)})^{\otimes M-1}$ for the configuration $c(2k-1) = -1$, $c(2k) = 1$ for $1 \leqslant k \leqslant M-2$, $c(2M-1) = -1$ in the following way. Notice that adjacent strings have the opposite directions. Following ref. 15, we introduce the following operators acting on $\mathscr{W}^{(0)}$:

$$Z_i = 1 \otimes \cdots \otimes Z \otimes \cdots \otimes 1 \tag{4.16}$$

$$X_i = 1 \otimes \cdots \otimes X \otimes \cdots \otimes 1 \tag{4.17}$$

where X and Z act on the ith factor $\mathbf{C}^N$ in $\mathscr{W}^{(0)}$ and are defined by

$$Z_{kl} = \delta_{k,l+1} \tag{4.18}$$

$$X_{kl} = \omega^k \delta_{k,l} \tag{4.19}$$

Table I. The Spectral Limits and the Resulting Boltzmann Weights and Yang–Baxter Operators

	Rapidities					Boltzmann weights (IRF-type R-matrix)	Yang–Baxter operators (braid group generators)
	p/q	q'/p	p'/q'	p'/q	spectral limits		
(Ia)	0	∞	1	∞	$p \ll q \ll p' = q'$	$\omega^{B(\delta,\beta) - B(\delta,\delta) + B(\alpha,\delta-\beta)}$	$T_k\, c(k) = 1 \quad \forall k$ left-regular representation
(Ib)	∞	∞	1	∞	$q \ll p \ll p' = q'$	$\omega^{-B(\beta,\delta) + B(\beta,\beta) + B(\alpha,\delta-\beta)}$	$T_k^{-1}\, c(k) = 1 \quad \forall k$ left-regular representation
(IIa)	1	0	0	0	$p' \ll q' \ll p = q$	$\omega^{B(\beta,\delta) - B(\beta,\beta) - B(\gamma,\delta-\beta)}$	$T_k\, c(k) = -1 \quad \forall k$ left-regular representation
(IIb)	1	0	∞	0	$q' \ll p' \ll p = q$	$\omega^{-B(\delta,\beta) + B(\delta,\delta) - B(\gamma,\delta-\beta)}$	$T_k^{-1}\, c(k) = -1 \quad \forall k$ left-regular representation
(IIIa)	1	∞	0	∞	$p = q \ll p' \ll q'$	$\omega^{B(\beta,\delta) - B(\delta,\delta) + B(\delta-\beta,\gamma)}$	$T_k\, c(k) = 1 \quad \forall k$ right-regular representation
(IIIb)	1	∞	∞	∞	$p = q \ll q' \ll p'$	$\omega^{-B(\delta,\beta) + B(\beta,\beta) + B(\delta-\beta,\gamma)}$	$T_k^{-1}\, c(k) = 1 \quad \forall k$ right-regular representation
(IVa)	0	0	1	0	$p' = q' \ll p \ll q$	$\omega^{B(\delta,\beta) - B(\beta,\beta) - B(\delta-\beta,\alpha)}$	$T_k\, c(k) = -1 \quad \forall k$ right-regular representation
(IVb)	∞	0	1	0	$p' = q' \ll q \ll p$	$\omega^{-B(\beta,\delta) + B(\delta,\delta) - B(\delta-\beta,\alpha)}$	$T_k^{-1}\, c(k) = -1 \quad \forall k$ right-regular representation
(Va)	0	0	0	0	$p' \ll q' \ll p \ll q$	$\sum_{\mu \in L} \omega^{B(\beta-\gamma,\alpha-\beta)} \omega^{-B(\mu,\mu) + B(\mu,\beta+\delta-\alpha-\gamma)}$	$T_{2k+1} T_{2k} T_{2k+2} T_{2k+1}^{-1} \quad c(2k-1) = -1,\ c(2k) = 1$
(Vb)	∞	0	∞	0	$q' \ll p' \ll q \ll p$	$\sum_{\mu \in L} \omega^{-B(\delta-\gamma,\alpha-\delta)} \omega^{B(\mu,\mu) + B(\mu,\beta+\delta-\alpha-\gamma)}$	$T_{2k+1} T_{2k}^{-1} T_{2k+2}^{-1} T_{2k+1}^{-1} \quad c(2k-1) = -1,\ c(2k) = 1$

(VIa)	0	∞	0	∞	$p \ll q \ll p' \ll q'$	$\sum_{\mu \in L} \omega^{B(\delta-\alpha,\gamma-\delta)} \omega^{-B(\mu,\mu)+B(\mu,\beta+\delta-\alpha-\gamma)}$	$T_{2k+1}^{-1} T_{2k} T_{2k+2} T_{2k+1}$	$c(2k-1) = -1, c(2k) = 1$
(VIb)	∞	∞	∞	∞	$q \ll p \ll q' \ll p$	$\sum_{\mu \in L} \omega^{-B(\beta-\alpha,\gamma-\beta)} \omega^{B(\mu,\mu)+B(\mu,\beta+\delta-\alpha-\gamma)}$	$T_{2k+1}^{-1} T_{2k}^{-1} T_{2k+2}^{-1} T_{2k+1}$	$c(2k-1) = -1, c(2k) = 1$
(VIIa)	0	∞	∞	∞	$p \ll q \ll q' \ll p'$	$D \prod_{i=1}^{n} \delta_{\beta_{i+1}-\gamma_{i+1},\alpha_i-\delta_i} \omega^{B(\delta,\gamma)-B(\alpha,\beta)}$	$T_{2k+1}^{-1} T_{2k} T_{2k+2}^{-1} T_{2k+1}$	$c(2k-1) = -1, c(k) = 1$
(VIIb)	∞	∞	0	∞	$q \ll p \ll p' \ll q'$	$D \prod_{i=1}^{n} \delta_{\delta_{i+1}-\gamma_{i+1},\alpha_i-\beta_i} \omega^{B(\beta-\alpha,\beta-\delta)}$	$T_{2k+1}^{-1} T_{2k}^{-1} T_{2k+2} T_{2k+1}$	$c(2k-1) = -1, c(k) = 1$
(VIIIa)	0	0	∞	0	$q' \ll p' \ll p \ll q$	$D \prod_{i=1}^{n} \delta_{\beta_{i+1}-\alpha_{i+1},\gamma_i-\delta_i} \omega^{B(\delta-\gamma,\delta-\beta)}$	$T_{2k+1} T_{2k} T_{2k+2}^{-1} T_{2k+1}^{-1}$	$c(2k-1) = -1, c(2k) = 1$
(VIIIb)	∞	0	0	0	$p' \ll q' \ll q \ll p$	$D \prod_{i=1}^{n} \delta_{\delta_{i+1}-\alpha_{i+1},\gamma_i-\beta_i} \omega^{B(\beta,\alpha)-B(\gamma,\delta)}$	$T_{2k+1} T_{2k}^{-1} T_{2k+2} T_{2k+1}^{-1}$	$c(2k-1) = -1, c(k) = 1$
(IXa)	0	∞	0	0	$p \ll p' \ll q \ll q'$	$D\delta_{\delta-\gamma,\alpha-\beta} \omega^{B(\alpha-\delta,\delta-\gamma)}$	$T_{2k+1} T_{2k} T_{2k+2} T_{2k+1}$	$c(2k-1) = -1, \quad c(2k) = 1$
(Xa)					$p' \ll p \ll q' \ll q$			
(IXb)	∞	0	∞	∞	$q \ll q' \ll p \ll p'$	$D\delta_{\delta-\gamma,\alpha-\beta} \omega^{-B(\alpha-\beta,\beta-\gamma)}$	$T_{2k+1}^{-1} T_{2k}^{-1} T_{2k+2}^{-1} T_{2k+1}^{-1}$	$c(2k-1) = -1, c(2k) = 1$
(Xb)					$q' \ll q \ll p' \ll p$			

for $k, l \in Z_N$. Moreover, using the operators (4.16) and (4.17), we define for $\alpha \in L$

$$X^{\alpha} = X_1^{\alpha_1} \cdots X_n^{\alpha_n} \tag{4.20}$$

$$Z^{\alpha} = Z_1^{\alpha_1} \cdots Z_n^{\alpha_n} \tag{4.21}$$

Then the representation $\mathscr{R}$ is given by

$$\mathscr{R}(x_{2k-1}^{\alpha}) = 1 \otimes \cdots \otimes Z^{\alpha} \otimes \cdots \otimes 1 \tag{4.22}$$

$$\mathscr{R}(x_{2k}^{\alpha}) = 1 \otimes \cdots \otimes X^{B\alpha} \otimes X^{-B\alpha} \otimes \cdots \otimes 1 \tag{4.23}$$

where the action of Z^{α} in (4.22) is on the kth space, while the action of $X^{B\alpha} \otimes X^{-B\alpha}$ in (4.23) is on the kth and $(k+1)$th spaces. Notice that it is possible to multiply the operators $T_{k+1}(s_k(c))\, T_k(c)$ relative to configurations which differ by a permutation, because the algebras $\mathscr{A}(c)$ arising from configurations of the same type are canonically isomorphic through (3.8) and (3.9). Thus, as a consequence, the matrix elements of the Yang–Baxter operators Y_{k0} in the limits (V)–(X) are exactly the matrix elements of the products

$$T_{\pm 2k+1}(s_{2k+1}(c))\, T_{\pm 2k}(s_{2k+1}(c))\, T_{\pm 2k+2}(s_{2k+1}(c))\, T_{\pm 2k+1}(c)$$

in the representation $\mathscr{R}$, where in (4.15) we have omitted to write the label $\mathscr{R}$.

Moreover, we have verified that the trace on the braid group representation given by the operators Y_{k0} in the limits (V) and (VI) enjoys the Markov properties. This can be verified immediately by observing that in the representation $\mathscr{R}$ the trace has the properties (3.19). Let us show, e.g., the invariance under the Markov move 2 in the case (Va). We define

$$\begin{aligned} \pi\colon \quad & B_{M-2} \to (\mathscr{W}^{(0)})^{M-1} \\ \pi(b_k) &= Y_{k0}, \qquad k = 1, ..., M-3 \end{aligned} \tag{4.24}$$

where the b_k are the braid group generators satisfying the relations

$$\begin{aligned} b_k b_{k'} &= b_{k'} b_k \qquad \text{for} \quad k, k' = 1, ..., M-3, \quad |k-k'| \geqslant 2 \\ b_k b_{k+1} b_k &= b_{k+1} b_k b_{k+1} \qquad \text{for} \quad k = 1, ..., M-4 \end{aligned} \tag{4.25}$$

and

$$\tau'(\hat{b}) = D^{M-3}\, \mathrm{Tr}(\pi(b)) \tag{4.26}$$

where the trace is normalized as $\mathrm{Tr}(1) = 1$. Indeed, omitting to write the configuration c on which the operators act, by applying repeatedly first and second Markov moves, as well as the braid group relations, we obtain

$$\begin{aligned}
\tau'(\widehat{gb_{M-1}}) &= D^{M-2}\,\mathrm{Tr}(\pi(g)\,T_{2M-1}T_{2M-2}T_{2M}T_{2M-1}^{-1}) \\
&= D^{M-2}\,\mathrm{Tr}(T_{2M-1}^{-1}\pi(g)\,T_{2M-1}T_{2M-2}T_{2M}) \\
&= D^{M-5/2}\,\mathrm{Tr}(T_{2M-1}^{-1}\pi(g)\,T_{2M-1}T_{2M-2}) \\
&= D^{M-5/2}\,\mathrm{Tr}(\pi(g)\,T_{2M-1}T_{2M-2}T_{2M-1}^{-1}) \\
&= D^{M-5/2}\,\mathrm{Tr}(\pi(g)\,T_{2M-2}^{-1}T_{2M-1}T_{2M-2}) \\
&= D^{M-5/2}\,\mathrm{Tr}(T_{2M-2}\pi(g)\,T_{2M-2}^{-1}T_{2M-1}) \\
&= D^{M-3}\,\mathrm{Tr}(T_{2M-2}\pi(g)\,T_{2M-2}^{-1}) \\
&= \tau'(\hat{g})
\end{aligned} \tag{4.27}$$

where $g \in B_{M-1}$ and $b_{M-1} \in B_M$.

If N is even, the operators Y_{k0} are well defined and it is possible to construct the representation π of the braid group. In the case (V) and (VI) it is possible to verify that the ordinary trace on the representation enjoys the Markov properties and hence we obtain the same link invariants that we have when N is odd. These properties can be verified directly on the representation π.

To summarize, we have shown that the ordinary trace on the Y_{k0} is invariant under the Markov moves 1 and 2, and hence provides tangle invariants (The tangles are in correspondence with the Yang–Baxter operators). We collect the results of this section in Table I.

5. GENERALIZATIONS

In the previous section we showed that the 3D Baxter–Bazhanov model can be related to the cyclotomic knot invariants generated by the limits (I)–(IV) of the associated Yang–Baxter operators Y_k. Under the other limits (V)–(X) one obtains products like

$$T_{\pm 2k+1}(s_{2k+1}(c))\,T_{\pm 2k}(s_{2k+1}(c))\,T_{\pm 2k+2}(s_{2k+1}(c))\,T_{\pm 2k+1}(c) \tag{5.1}$$

It is intriguing to think that the products (5.1) give a "cabling" representation of the braid group, analogously to the procedure established by Akutsu, Wadati, and Deguchi[13,14] to construct higher-dimensional braid group representations of the Hecke algebra of B_M. However some observations are in order:

(i) The cabling here is perhaps related to higher-dimensional representations of $U_q(\widehat{gl}(n))$ with $q^N = 1$.

(ii) Probably we must give up the orientation, and hence the invariants are of nonoriented type.

(iii) The single T_k are related to a representation of the Temperley–Lieb algebra[17,18] only for $N = 2$, 3 and $n = 2$. Therefore only in these cases one may think to generalize the construction implemented in refs. 13 and 14.

Work along this direction is in progress.

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